M427J HW 9 Solutions

1 Section 3.4

1.

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$$

The eigenvalues of A are $\lambda = 4, 2$; the eigenvector corresponding to $\lambda_1 = 4$ is $\xi_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

The eigenvector corresponding to $\lambda_2 = 2$ is $\xi_2 = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$. The norms of the eigenvectors are $\|\xi_1\| = \sqrt{2}$ and $\|\xi_2\| = \sqrt{10}$, so that the corresponding normalized eigenvectors are

$$\widehat{\xi_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \widehat{\xi_2} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

(the hat-notation $\widehat{\cdot}$ denotes a unit-vector).

2.

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

The eigenvalue/vector pairs are:

$$\lambda_1 = 1 + 2i, \;\; \xi_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

and

$$\lambda_2 = 1 - 2i, \;\; \xi_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}.$$

After normalizing by $\|\xi_1\| = \|\xi_2\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$, we get

$$\widehat{\xi_1} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1+i \\ 2 \end{bmatrix}, \quad \widehat{\xi_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1-i \\ 2 \end{bmatrix}.$$

3.

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

The eigenvalue/vector pairs are:

$$\lambda_1 = -1, \;\; \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\lambda_2 = -3, \quad \xi_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

so that $\|\xi_1\| = \|\xi_2\| = \sqrt{2}$. The normalized eigenvectors then are

$$\widehat{\xi_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \widehat{\xi_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

2 Section 3.5

1.

$$\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \mathbf{x}$$

The general solution the first order ODE system is

$$\mathbf{x}_{\text{gen}} = c_1 e^{\lambda_1 t}\ \xi_1 + c_2 e^{\lambda_2 t}\ \xi_2$$

where λ_i, ξ_i are the eigenvalue/eigenvector pairs of the matrix above. They are

$$\lambda_1 = -1,\ \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\lambda_2 = -2,\ \xi_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Note there is no reason to normalize here, since c_1, c_2 are arbitrary until fixed by an initial condition.

2.

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x}$$

The general solution the first order ODE system is

$$\mathbf{x}_{\text{gen}} = c_1 e^{\lambda_1 t}\ \xi_1 + c_2 e^{\lambda_2 t}\ \xi_2$$

where λ_i, ξ_i are the eigenvalue/eigenvector pairs of the matrix above. They are

$$\lambda_1 = 1,\ \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\lambda_2 = -1,\ \xi_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

3.

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x}$$

The general solution the first order ODE system is

$$\mathbf{x}_{\text{gen}} = c_1 e^{\lambda_1 t}\, \xi_1 + c_2 e^{\lambda_2 t}\, \xi_2$$

where λ_i, ξ_i are the eigenvalue/eigenvector pairs of the matrix above. They are

$$\lambda_1 = 2,\ \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\lambda_2 = -3,\ \xi_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

5.

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Now we have an IVP. The general solution is, again, given by the linear combination of eigenvectors/eigenvalues of the form:

$$\mathbf{x}_{\text{gen}} = c_1 e^{\lambda_1 t}\, \xi_1 + c_2 e^{\lambda_2 t}\, \xi_2.$$

The eigenvector/eigenvalue pairs are

$$\lambda_1 = -1,\ \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\lambda_2 = 3,\ \xi_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

which gives

$$\mathbf{x}_{\text{gen}} = c_1\, e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\, e^{3t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Now we use the initial condition to solve for c_1, c_2:

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

The solution to the two-equation, two-unknown system is $c_1 = 1/2$ and $c_2 = 1/2$, so that

$$\mathbf{x} = \frac{1}{2}\, e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\, e^{3t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

As $t \to \infty$, $(11/4)\, e^{-t} \to 0$, while $(1/2)\, e^{3t} \to +\infty$, so that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} +\infty \\ +\infty \end{bmatrix}$$

as $t \to \infty$.

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